

Package ‘freealg’

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Type Package

Title The Free Algebra

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Maintainer Robin K. S. Hankin <hankin.robin@gmail.com>

Depends R (>= 3.5.0), methods

Description The free algebra in R; multivariate polynomials with non-commuting indeterminates.

License GPL (>= 2)

LazyData yes

Imports Rcpp (>= 1.0-7), partitions (>= 1.9-22), mathjaxr, disordR (>= 0.0-8)

LinkingTo Rcpp

SystemRequirements C++11

Suggests knitr, testthat, magrittr, markdown, rmarkdown

VignetteBuilder knitr

URL <https://github.com/RobinHankin/freealg>

BugReports <https://github.com/RobinHankin/freealg/issues>

RdMacros mathjaxr

R topics documented:

freealg-package	2
accessor	3
constant	4
deriv	5
dot-class	7
freealg	8
horner	9
linear	10
Ops.freealg	11
pepper	12
print	13
rfalg	14
subs	15
zero	16
Index	17

freealg-package

*The Free Algebra***Description**

The free algebra in R; multivariate polynomials with non-commuting indeterminates.

Details

The DESCRIPTION file:

```

Package:          freealg
Type:             Package
Title:           The Free Algebra
Version:          1.0-4
Authors@R:       person(given=c("Robin", "K. S."), family="Hankin", role = c("aut","cre"), email="hankin.robin@
Maintainer:      Robin K. S. Hankin <hankin.robin@gmail.com>
Depends:         R (>= 3.5.0), methods
Description:     The free algebra in R; multivariate polynomials with non-commuting indeterminates.
License:         GPL (>= 2)
LazyData:        yes
Imports:         Rcpp (>= 1.0-7), partitions (>= 1.9-22), mathjaxr, disordR (>= 0.0-8)
LinkingTo:       Rcpp
SystemRequirements: C++11
Suggests:        knitr,testthat,magrittr,markdown,rmarkdown
VignetteBuilder: knitr
URL:             https://github.com/RobinHankin/freealg
BugReports:      https://github.com/RobinHankin/freealg/issues
RdMacros:        mathjaxr
Author:          Robin K. S. Hankin [aut, cre] (<https://orcid.org/0000-0001-5982-0415>)

```

Index of help topics:

Ops.freealg	Arithmetic Ops methods for the the free algebra
accessors	Accessor methods for freealg objects
constant	The constant term
deriv	Differentiation of 'freealg' objects
dot-class	Class "dot"
freealg	The free algebra
freealg-package	The Free Algebra
horner	Horner's method
linear	A simple free algebra object
pepper	Combine variables in every possible order
print.freealg	Print freealg objects
rfalg	Random free algebra objects
subs	Substitution
zero	The zero algebraic object

Author(s)

NA

Maintainer: Robin K. S. Hankin <hankin.robin@gmail.com>

Examples

```

a <- as.freealg("x+xyx")
b <- as.freealg("4x +XyX") # upper-case interpreted as inverse

a+b
stopifnot(a+b==b+a) # should be TRUE

a*b ==b*a # FALSE; noncommutative algebra

as.freealg("1+X+xy")^3

rfalg()
rfalg()^2

```

 accessor

Accessor methods for freealg objects

Description

Accessor methods for free algebra objects

Usage

```

words(x)
coeffs(x)
coeffs(x) <- value

```

Arguments

x	Object of class freealg
value	Numeric vector of length 1

Details

Access or set the different parts of an freealg object. The constant term is technically a coefficient but is documented under constant.Rd.

Note

There is an extended discussion of this issue in the mvp object at accessor.Rd.

Author(s)

Robin K. S. Hankin

See Also

[constant](#)

Examples

```
a <- rfalg()
coeffs(a)
coeffs(a) <- 7
```

constant

The constant term

Description

Get and set the constant term of a freealg object

Usage

```
## S3 method for class 'freealg'
constant(x)
## S3 method for class 'numeric'
constant(x)
## S3 replacement method for class 'freealg'
constant(x) <- value
is.constant(x)
```

Arguments

x	Object of class freealg
value	Scalar value for the constant

Details

The constant term in a free algebra object is the coefficient of the empty term. In a freealg object, the map including $\{\} \rightarrow v$ implies that v is the constant.

If x is a freealg object, `constant(x)` returns the value of the constant in the multivariate polynomial; if x is numeric, it returns a constant freealg object with value x .

Function `is.constant()` returns TRUE if its argument has no variables and FALSE otherwise.

Setting the coefficients of the empty freealg returns the zero (empty) object.

Author(s)

Robin K. S. Hankin

Examples

```
p <- as.freealg("1+X+Y+xy")

constant(p)
constant(p^5)

constant(p) <- 1000
p
```

 deriv

Differentiation of freealg objects

Description

Differentiation of freealg objects

Usage

```
## S3 method for class 'freealg'
deriv(expr, r, ...)
```

Arguments

expr	Object of class freealg
r	Integer vector. Elements denote variables to differentiate with respect to. If r is a character vector, it is interpreted as a=1, b=2, ..., z=26; if of length 1, "aab" is interpreted as c("a", "a", "b")
...	Further arguments, currently ignored

Details

Experimental function `deriv(S, v)` returns $\frac{\partial^r S}{\partial v_1 \partial v_2 \dots \partial v_r}$. The Leibniz product rule

$$(u \cdot v)' = uv' + u'v$$

operates even if (as here) u, v do not commute. For example, if we wish to differentiate $aaba$ with respect to a , we would write $f(a) = aaba$ and then

$$f(a + \delta a) = (a + \delta a)(a + \delta a)b(a + \delta a)$$

and working to first order we have

$$f(a + \delta a) - f(a) = (\delta a)aba + a(\delta a)ba + aab(\delta a).$$

In the package:

```
> deriv(as.freealg("aaba"), "a")
free algebra element algebraically equal to
+ 1*aab(da) + 1*a(da)ba + 1*(da)aba
```

A term of a freealg object can include negative values which correspond to negative powers of variables. Thus:

```
> deriv(as.freealg("AAAA"), "a")
free algebra element algebraically equal to
- 1*AAAA(da)A - 1*AAA(da)AA - 1*AA(da)AAA - 1*A(da)AAAA
```

(see also the examples). Vector *r* may include negative integers which mean to differentiate with respect to the inverse of the variable:

```
> deriv(as.freealg("3abcbCC"), "C")
free algebra element algebraically equal to
+ 3*abcbC(dC) + 3*abcb(dC)C - 3*abc(dC)cbCC
```

It is possible to perform repeated differentiation by passing a suitable value of *r*. For $\frac{\partial^2}{\partial a \partial c}$:

```
> deriv(as.freealg("aabaAcx"), "ac")
free algebra element algebraically equal to
- 1*aabaA(da)A(dc)x + 1*aa(da)ba(dc)x + 1*a(da)abA(dc)x + 1*(da)aabA(dc)x
```

The infinitesimal indeterminates (“da” etc) are represented by SHRT_MAX+r, where *r* is the integer for the symbol, and SHRT_MAX is the maximum short integer. This includes negative *r*. So the maximum number for any symbol is SHRT_MAX. Inverse elements such as A, being represented by negative integers, have differentials that are SHRT_MAX-r.

Function `deriv()` calls helper function `lowlevel_diffn()` which is documented at `Ops.freealg.Rd`.

A vignette illustrating this concept and furnishing numerical verification of the code in the context of matrix algebra is given at `inst/freealg_matrix.Rmd`.

Author(s)

Robin K. S. Hankin

Examples

```
deriv(as.freealg("4*aaaabaacAc"), 1)

x <- rfa1g()
deriv(x, 1:3)

y <- rfa1g(7, 7, 17, TRUE)

deriv(y, 1:5) - deriv(y, sample(1:5)) # should be zero
```

dot-class

Class "dot"

Description

The dot object is defined so that idiom like `. [x,y]` returns the commutator, that is, $xy-yx$ or the Lie bracket $[x,y]$. It would have been nice to use `[x,y]` (that is, without the dot) but although this is syntactically consistent, it cannot be done in R.

The "meat" of the package is:

```
setClass("dot", slots = c(ignore='numeric'))
`.` <- new("dot")
setMethod("[", signature(x="dot", i="ANY", j="ANY"), function(x,i,j,drop){i*j-j*i})
```

The package code includes other bits and pieces such as informative error messages for idiom such as `. []`. The package defines a matrix method for the dot object. This is because `*` returns (incorrectly, in my view) the elementwise product, not the matrix product.

The Jacobi identity, satisfied by any associative algebra, is

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

and the left hand side is returned by `jacobi()`, which should be zero (for some definition of "zero").

The dot object is generated by running script `inst/dot.Rmd`, which includes some further discussion and technical documentation, and creates file `dot.rda` which resides in the `data/` directory.

Value

Always returns an object of the same class as `xy`.

Author(s)

Robin K. S. Hankin

Examples

```
x <- rfalg()
y <- rfalg()
z <- rfalg()

jacobi(x,y,z) # Jacobi identity
.`[x,.`[y,z]] + .`[y,.`[z,x]] + .`[z,.`[x,y]] # Jacobi, expanded

rM <- function(...){matrix(sample(1:9,9),3,3)}

.`[rM(),rM()]
jacobi(rM(),rM(),rM())
```

freealg

*The free algebra***Description**

Create, test for, and coerce to, freealg objects

Usage

```
freealg(words, coeffs)
is_ok_free(words, coeffs)
is.freealg(x)
as.freealg(x, ...)
char_to_freealg(ch)
natural_char_to_freealg(string)
string_to_freealg(string)
vector_to_free(v, coeffs)
```

Arguments

words	Terms of the algebra object, eg $c(1, 2, -1, -3, -2)$ corresponds to $abACB$ because $a = 1, b = 2$ etc; uppercase, or negative number, means inverse
coeffs	Numeric vector corresponding to the coefficients of each element of the word list
string	Character string
ch	Character vector
v	Vector of integers
x	Object possibly of class freealg
...	Further arguments, passed to the methods

Details

Function `freealg()` is the formal creation mechanism for freealg objects. However, it is not very user-friendly; it is better to use `as.freealg()` in day-to-day use.

Function `is_ok_freealg()` checks for consistency of its arguments.

A freealg object is a two-element list. The first element is a list of integer vectors representing the indices and the second is a numeric vector of coefficients. Thus, for example:

```
> as.freealg("a+4bd+3abbbbc")
free algebra element algebraically equal to
+ 1*a + 3*abbbbc + 4*bd
> dput(as.freealg("a+4bd+3abbbbc"))
structure(list(indices = list(1L, c(1L, 2L, 2L, 2L, 2L, 3L),
  c(2L, 4L)), coeffs = c(1, 3, 4)), class = "freealg")
```

Observe that the order of the terms is not preserved and indeed is undefined (implementation-specific). Zero entries are stripped out.

Character strings may be coerced to `freealg` objects; as `.freealg()` calls `natural_char_to_freealg()`, which is user-friendly. Functions `char_to_freealg()` and `string_to_freealg()` are low-level helper functions. These functions assume that upper-case letters are the multiplicative inverses of the lower-case equivalents; so for example as `.freealg("aA")` and as `.freealg(aBcCbA)` evaluate to one. This can be confusing with the default print method.

Even though individual symbols have multiplicative inverses, a general element of the free algebra will not have a multiplicative inverse. For example, $1+x$ does not have an inverse. The free algebra is not a division algebra, in general.

Note

Internally, the package uses signed integers and as such can have `.Machine$integer.max` different symbols; on my machine this is 2147483647. Of course the print method cannot deal with this as it only has 26 symbols for letters a-z (and A-Z for the inverses), but the objects themselves do not care about the print method. Note also that the experimental calculus facility (as per `deriv()`) reserves numbers in the range $\text{SHRT_MAX} \pm r$ for infinitesimals, where r is the integer for a symbol. This system might change in the future.

Author(s)

Robin K. S. Hankin

Examples

```
freealg(list(1:2, 2:1, numeric(0), 1:6), 1:4)

freealg(sapply(1:5, seq_len), 1:5)

freealg(replicate(5, sample(-5:5, rgeom(1, 1/5), replace=TRUE)), 1:5)

as.freealg("1+xaX")^5
```

horner

Horner's method

Description

Horner's method for multivariate polynomials

Usage

```
horner(P, v)
```

Arguments

P	Free algebra polynomial
v	Numeric vector of coefficients

Details

This function is (almost) the same as `mvp : horner()`.

Given a polynomial

$$p(x) = a_0 + a_1 + a_2x^2 + \cdots + a_nx^n$$

it is possible to express $p(x)$ in the algebraically equivalent form

$$p(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + xa_n) \cdots))$$

which is much more efficient for evaluation, as it requires only n multiplications and n additions, and this is optimal. Function `horner()` will take a `freealg` object for its first argument.

Author(s)

Robin K. S. Hankin

Examples

```
horner("x", 1:4) # note constant term is 1.
```

```
horner("x+y",1:3) # note presence of xy and yx terms
```

```
horner("1+x+xyX",1:3)
```

linear

A simple free algebra object

Description

Create simple free algebra objects including linear expressions, for example

```
> linear(1:3)
free algebra element algebraically equal to
+ 1*a + 2*b + 3*c
> linear(1:3,power=5)
free algebra element algebraically equal to
+ 1*aaaaa + 2*bbbbb + 3*ccccc
>
```

Usage

```
linear(x,power=1)
```

Arguments

x	Numeric vector of terms
power	Integer vector of powers

Note

Many of the functions documented at `mvp::special.Rd` do not make sense in the context of the free algebra. Function `mvp::product()`, for example, imposes an order on the expansion.

Function `constant()` is documented at `constant.Rd`, but is listed below for convenience.

Author(s)

Robin K. S. Hankin

See Also

[constant](#), [zero](#)

Examples

```
linear(1:3)
linear(1:3,power=5)
linear(1:3,power=3:1)
```

Ops.freealg

Arithmetic Ops methods for the the free algebra

Description

Arithmetic operators for manipulation of freealg objects such as addition, multiplication, powers, etc

Usage

```
## S3 method for class 'freealg'
Ops(e1, e2)
free_negative(S)
free_power_scalar(S,n)
free_eq_free(e1,e2)
free_plus_numeric(S,x)
free_plus_free(e1,e2)
lowlevel_simplify(words,coeffs)
lowlevel_free_prod(words1,coeffs1,words2,coeffs2)
lowlevel_free_sum(words1,coeffs1,words2,coeffs2)
lowlevel_free_power(words,coeffs,n)
lowlevel_diffn(words,coeffs,r)
lowlevel_subs(words1, coeffs1, words2, coeffs2, r)
```

Arguments

S,e1,e2	Objects of class freealg
n	Integer, possibly non-positive
r	Integer vector indicating variables to differentiate with respect to
x	Scalar value

```
words, words1, words2
    A list of words, that is, a list of integer vectors representing the variables in each
    term
coeffs, coeffs1, coeffs2
    Numeric vector representing the coefficients of each word
```

Details

The function `Ops.freealg()` passes binary arithmetic operators (“+”, “-”, “*”, “^”, and “==”) to the appropriate specialist function.

The caret, as in a^n , denotes arithmetic exponentiation, as in $x^3 == x*x*x$.

Functions `lowlevel_foo()` are low-level functions that interface directly with the C routines in the `src/` directory and are not intended for the end-user.

Author(s)

Robin K. S. Hankin

Examples

```
rfalg()
as.freealg("1+x+xy+yx") # variables are non-commutative
as.freealg("x") * as.freealg("X") # upper-case letters are lower-case inverses
```

```
constant(as.freealg("x+y+X+Y")^6) # OEIS sequence A035610
```

pepper

Combine variables in every possible order

Description

Given a list of variables, construct every term comprising only those variables; function `pepper()` returns a free algebra object equal to the sum of these terms.

The function is named for a query from an exam question set by Sarah Marshall in which she asked how many ways there are to arrange the letters of word “pepper”, the answer being $\binom{6}{123} = \frac{6!}{1!2!3!} = 60$.

Function `multiset()` in the `partitions` package gives related functionality.

Usage

```
pepper(v)
```

Arguments

`v` Variables to combine. If a character string, coerce to variable numbers

Author(s)

Robin K. S. Hankin

See Also[linear](#)**Examples**

```
pepper(c(1,2,2,2,3))
pepper("pepper")
```

print	<i>Print freealg objects</i>
-------	------------------------------

Description

Print methods for free algebra objects

Usage

```
## S3 method for class 'freealg'
print(x,...)
```

Arguments

x	Object of class freealg in the print method
...	Further arguments, currently ignored

Note

The print method does not change the internal representation of a freealg object, which is a two-element list, the first of which is a list of integer vectors representing words, and the second is a numeric vector of coefficients.

The print method has special dispensation for length-zero freealg objects but these are not handled entirely consistently.

The print method is sensitive to the value of `getOption("usecaret")`, defaulting to "no". The default is to use uppercase letters to represent multiplicative inverses, but if TRUE, inverses are indicated using "⁻¹". This becomes cumbersome for powers above the first. For example, the default notation for aba^{-2} is abAA but becomes $aba^{-1}a^{-1}$ if usecaret is TRUE.

Integers exceeding SHRT_MAX are reserved for infinitesimals, which are printed as "da"; see the note at `deriv.Rd` for details.

Author(s)

Robin K. S. Hankin

See Also[freealg.deriv](#)

Examples

```

rfalg()

x <- rfalg(inc=TRUE)
x                                # default
options("usecaret" = TRUE) # use caret
x
options("usecaret" = FALSE) # back to the default
x

```

rfalg

Random free algebra objects

Description

Random elements of the free algebra, intended as quick “get you going” examples of freealg objects

Usage

```
rfalg(n=7, distinct=3, maxsize=4, include.negative=FALSE)
```

Arguments

n	Number of terms to generate
distinct	Number of distinct symbols to use
maxsize	Maximum number of symbols in any word
include.negative	Boolean, with default FALSE meaning to use only positive symbols (lower-case letters) and TRUE meaning to use upper-case letters as well, corresponding to the inverse of the lower-case symbols

Details

What you see is what you get, basically. A term such as aaBaAbaC will be simplified to aaaC.

Author(s)

Robin K. S. Hankin

Examples

```

rfalg()
rfalg()^3

constant(rfalg())

```

subs	<i>Substitution</i>
------	---------------------

Description

Substitute symbols in a freealg object for numbers or other freealg objects

Usage

```
subs(S, ...)
subsu(S1, S2, r)
```

Arguments

S, S1, S2	Objects of class freealg
r	Integer specifying symbol to substitute ($a = 1, b = 2$ etc)
...	named arguments corresponding to variables to substitute

Details

Function subs() substitutes variables for freealg objects (coerced if necessary) using a natural R idiom. Observe that this type of substitution is sensitive to order:

```
> subs("ax", a="1+x", x="1+a")
free algebra element algebraically equal to
+ 2 + 3*a + 1*aa

> subs("ax", x="1+a", a="1+x")
free algebra element algebraically equal to
+ 2 + 3*x + 1*xx
```

Functions subsu() is a lower-level formal function, not really intended for the end-user. Function subsu() takes S1 and substitutes occurrences of symbol r with S2.

No equivalent to mvp: : subvec() is currently implemented.

Value

Returns a freealg object.

Author(s)

Robin K. S. Hankin

Examples

```

subs("abccc",b="1+3x")
subs("aaaa",a="1+x") # binomial

subs("abA",b=31)

subs("1+a",a="A") # can substitute for an inverse
subs("A",a="1+x") # inverses are not substituted for

## Sequential substitution works:

subs("abccc",b="1+3x",x="1+d+2e")
subs(rfalg(),a=rfalg())

```

zero

The zero algebraic object

Description

Test for a freealg object's being zero

Usage

```
is.zero(x)
```

Arguments

x Object of class freealg

Details

Function `is.zero()` returns TRUE if `x` is indeed the zero free algebra object. It is defined as `length(coeffs(x))==0` for reasons of efficiency, but conceptually it returns `x==constant(0)`.

(Use `constant(0)` to create the zero object).

Author(s)

Robin K. S. Hankin

See Also

[constant](#)

Examples

```
stopifnot(is.zero(constant(0)))
```


Index

- * **package**
 - freealg-package, 2
- * **symbolmath**
 - deriv, 5
 - horner, 9
 - linear, 10
 - pepper, 12
 - subs, 15
 - zero, 16
- . (dot-class), 7
- [, dot, ANY, ANY-method (dot-class), 7
- [, dot, ANY, missing-method (dot-class), 7
- [, dot, matrix, matrix-method (dot-class), 7
- [, dot, missing, ANY-method (dot-class), 7
- [, dot, missing, missing-method (dot-class), 7
- [, dot-method (dot-class), 7
- [.dot (dot-class), 7
- accessor, 3
- accessors (accessor), 3
- aderiv (deriv), 5
- as.freealg (freealg), 8
- char_to_freealg (freealg), 8
- coefficients (accessor), 3
- coeffs (accessor), 3
- coeffs<- (accessor), 3
- commutator (dot-class), 7
- constant, 4, 4, 11, 16
- constant<- (constant), 4
- deriv, 5, 13
- deriv_freealg (deriv), 5
- dot (dot-class), 7
- dot-class, 7
- dot_error (dot-class), 7
- extract (dot-class), 7
- free_eq_free (Ops.freealg), 11
- free_equal_free (Ops.freealg), 11
- free_negative (Ops.freealg), 11
- free_plus_free (Ops.freealg), 11
- free_plus_numeric (Ops.freealg), 11
- free_power_scalar (Ops.freealg), 11
- free_times_free (Ops.freealg), 11
- free_times_scalar (Ops.freealg), 11
- freealg, 8, 13
- freealg-package, 2
- freealg_negative (Ops.freealg), 11
- horner, 9
- is.constant (constant), 4
- is.freealg (freealg), 8
- is.zero (zero), 16
- is_ok_free (freealg), 8
- jacobi (dot-class), 7
- linear, 10, 13
- lowlevel_deriv (Ops.freealg), 11
- lowlevel_diff (Ops.freealg), 11
- lowlevel_diffn (Ops.freealg), 11
- lowlevel_free_power (Ops.freealg), 11
- lowlevel_free_prod (Ops.freealg), 11
- lowlevel_free_sum (Ops.freealg), 11
- lowlevel_simplify (Ops.freealg), 11
- lowlevel_subs (Ops.freealg), 11
- namechanger (subs), 15
- natural_char_to_freealg (freealg), 8
- numeric_to_free (freealg), 8
- ops (Ops.freealg), 11
- Ops.freealg, 11
- pepper, 12
- print, 13
- rvalg, 14
- rfreealg (rvalg), 14
- string_to_freealg (freealg), 8
- subs, 15
- substitute (subs), 15
- subs (subs), 15
- vector_to_free (freealg), 8

words (accessor), [3](#)

zero, [11](#), [16](#)